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Note

Explicit Energy-Conserving Schemes for the Three-Body Problem

1. The Problem

Conservation and symmetry are two fundamental characteristics of physics [1]. But ordinary schemes for computation do not guarantee conservation of total energy. Even for the simplest case of the system,

$$\frac{d^2x}{dt^2} = f(x),$$

the ordinary methods, including the Euler method and the Runge–Kutta method, do not conserve the total energy. This question was raised first by D. Greenspan [2] and implicit schemes were given. C. Qin [3] gave an explicit scheme for this system. It is natural to extend the method to the famous three-body problem.

2. The Explicit Energy-Conserving Scheme for the Three-Body Problem

Suppose there are three particles P_0, P_1, and P_2. Let M_j ($j=0, 1, 2$), (x_j, y_j, z_j) and (u_j, v_j, w_j) denote their masses, positions, and velocities at the time t, respectively. Denote the three distances by

$$R_0 = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{(1/2)},$$
$$R_1 = [(x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2]^{(1/2)},$$
$$R_2 = [(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2]^{(1/2)}.$$

Then the potential of this system can be written in the form [4]:

$$V = V(x, y, z) = -\left\{\frac{M_0 M_1}{R_2} + \frac{M_1 M_2}{R_0} + \frac{M_2 M_0}{R_1}\right\}. \tag{1}$$

Here the gravitational constant G is reduced to unity.

The system of nine equations describing the motion of the three bodies is

$$M_j \frac{d^2 x_j}{dt^2} = -\frac{\delta V}{\delta x_j}, \qquad M_j \frac{d^2 y_j}{dt^2} = -\frac{\delta V}{\delta y_j}, \qquad M_j \frac{d^2 z_j}{dt^2} = -\frac{\delta V}{\delta z_j} \tag{2}$$

($j = 0, 1, 2$).

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Or in the form of a system of 18 ordinary differential equations:

$$
\begin{aligned}
&\frac{dx_j}{dt}=u_j, \qquad \frac{dy_j}{dt}=v_j, \qquad \frac{dz_j}{dt}=w_j,\\
&M_j\frac{du_j}{dt}=-\frac{\delta V}{\delta x_j}, \qquad M_j\frac{dv_j}{dt}=-\frac{\delta V}{\delta y_j}, \qquad M_j\frac{dw_j}{dt}=-\frac{\delta V}{\delta z_j}
\end{aligned} \tag{3}
$$

$(j=0, 1, 2)$.

Denote the total kinetic energy of the system by Kin:

$$\text{Kin}=\text{Kin}(u, v, w)=\frac{1}{2}\sum_{j=0}^{2} M_j(u_j^2+v_j^2+w_j^2). \tag{4}$$

The equation of conservation of energy is then:

$$\text{Kin}(u, v, w)+V(x, y, z)=E. \tag{5}$$

Here E is the total energy of this system to be conserved.

When we take the center of gravity of this system as the origin of the coordinate system, then we have also three identities,

$$\sum_{j=0}^{2} M_j x_j=\sum_{j=0}^{2} M_j y_j=\sum_{j=0}^{2} M_j z_j=0, \tag{6}$$

and three equations of conservation of linear momentums,

$$\sum_{j=0}^{2} M_j u_j=\sum_{j=0}^{2} M_j v_j=\sum_{j=0}^{2} M_j w_j=0. \tag{7}$$

To any three fixed coordinate axes, there are three equations of conservation of angular momentum:

$$
\begin{aligned}
&\sum_{j=0}^{2} M_j(y_j w_j-z_j v_j)=a,\\
&\sum_{j=0}^{2} M_j(z_j u_j-x_j w_j)=b,\\
&\sum_{j=0}^{2} M_j(x_j v_j-y_j u_j)=c.
\end{aligned} \tag{8}
$$

In this section we propose an explicit scheme for conserving the total energy for the case $E<0$. For the case $E\geqslant 0$ we shall propose in the next section an explicit scheme to conserve not only the total energy but also the angular momentums.

For the case $E<0$, we take two steps: first we take an approximation; then we correct it in order to conserve the total energy E.

Take an approximation formula:

$$
\begin{aligned}
\bar{x}_j^{-(n+1)} &= x_j^{(n)} + \tau u_j^{(n)} - \frac{(\tau)^2}{2M_j}\frac{\delta V^{(n)}}{\delta x_j},\\
\bar{y}_j^{-(n+1)} &= y_j^{(n)} + \tau v_j^{(n)} - \frac{(\tau)^2}{2M_j}\frac{\delta V^{(n)}}{\delta y_j},\\
\bar{z}_j^{-(n+1)} &= z_j^{(n)} + \tau w_j^{(n)} - \frac{(\tau)^2}{2M_j}\frac{\delta V^{(n)}}{\delta z_j},\\
\bar{u}_j^{-(n+1)} &= u_j^{(n)} - \tau \frac{1}{M_j}\frac{\delta V^{(n)}}{\delta x_j},\\
\bar{v}_j^{-(n+1)} &= v_j^{(n)} - \tau \frac{1}{M_j}\frac{\delta V^{(n)}}{\delta y_j},\\
\bar{w}_j^{-(n+1)} &= w_j^{(n)} - \tau \frac{1}{M_j}\frac{\delta V^{(n)}}{\delta z_j}
\end{aligned} \tag{9}
$$

$(j=0,\ 1,\ 2)$.

Here τ is the time step. (9) is a part of the ordinary Taylor expansions and does not satisfy condition (5). In order that (5) will be satisfied, we modify the first three quantities by a factor σ:

$$
\begin{aligned}
x_j^{(n+1)} &= \sigma\, \bar{x}_j^{-(n+1)},\\
y_j^{(n+1)} &= \sigma\, \bar{y}_j^{-(n+1)},\\
z_j^{(n+1)} &= \sigma\, \bar{z}_j^{-(n+1)},\\
u_j^{(n+1)} &= \bar{u}_j^{-(n+1)},\\
v_j^{(n+1)} &= \bar{v}_j^{-(n+1)},\\
w_j^{(n+1)} &= \bar{w}_j^{-(n+1)}.
\end{aligned} \tag{10}
$$

Substituting (10) and (9) into (5) to determine the factor σ, one gets:

$$
\mathrm{Kin}(u^{(n+1)}, v^{(n+1)}, w^{(n+1)}) + \frac{1}{\sigma} V(\bar{x}^{-(n+1)}, \bar{y}^{-(n+1)}, \bar{z}^{-(n+1)}) = E,
$$

or

$$
\sigma = \frac{V(\bar{x}^{-(n+1)}, \bar{y}^{-(n+1)}, \bar{z}^{-(n+1)})}{E - \mathrm{Kin}(u^{(n+1)}, v^{(n+1)}, w^{(n+1)})}. \tag{11}
$$

Since we have assumed that $E<0$, and of course, $\mathrm{Kin}^{(n+1)} \geqslant 0$, $V^{(n+1)} \geqslant 0$, $V^{(n+1)} < 0$, so the denominator is not zero and σ is uniquely determined by (11).

The scheme runs then in the order of (9), (11), and (10). It is explicit and it satisfies condition (5).

It is easy to verify that, due to the property of the Newtonian potential $V(x, y, z)$, the relations (6) and (7) are also conserved, once the initial conditions are satisfied.

Since very few exact solutions of the three-body problem are known, we can only take the famous Lagrange triangular configuration as an example to compare with the computational results.

Take the masses of the three points as

$$M_0 = M_1 = M_2 = \sqrt{3},$$

and their initial positions and initial velocities as follows:

$$\begin{aligned}
(x_0, y_0, z_0) &= (0, 1, 0),\\
(x_1, y_1, z_1) &= (-\sqrt{3}/2, -1/2, 0),\\
(x_2, y_2, z_2) &= (\sqrt{3}/2, -1/2, 0),\\
(u_0, v_0, w_0) &= (-1, 0, 0),\\
(u_1, v_1, w_1) &= (1/2, -\sqrt{3}/2, 0),\\
(u_2, v_2, w_2) &= (1/2, \sqrt{3}/2, 0).
\end{aligned}$$

Then (6) and (7) are satisfied initially, and the initial Kin and V are

$$\text{Kin} = 3\sqrt{3}/2, \qquad V = -3\sqrt{3},$$

so that the total energy

$$E = -3\sqrt{3}/2 < 0.$$

The lengths of the sides of the triangular configuration are $\sqrt{3}$ and should be the same at all times.

Now use the explicit energy conserving scheme with time increment $\tau = 0.001$ the calculations are carried out and the results of the three sides at the time $n\tau$ are given in Table I.

TABLE I

n	$\overline{P_0P_1}$	$\overline{P_1P_2}$	$\overline{P_2P_0}$
0	1.73205081	1.73205081	1.73205081
10	1.73205074	1.73205081	1.73205074
20	1.73205019	1.73205081	1.73205020
30	1.73204864	1.73205082	1.73204872
40	1.73204558	1.73205083	1.73204582
50	1.73204046	1.73205084	1.73204174
60	1.73203274	1.73205087	1.73203396
70	1.73202187	1.73205090	1.73202413
80	1.73200728	1.73205094	1.73201115
90	1.73198841	1.73205100	1.73199461
100	1.73196467	1.73205106	1.73197413

3. An Explicit All Conserving Scheme for the Three-Body Problem

The scheme in last section does not conserve the angular momentums. In case that the total energy E is non-negative, we propose in this section an explicit scheme conserving all the 10 relations, i.e. (5), (6), (7), and (8).

The idea runs as follows: We still take the first approximation (9), but change (10) as follows:

$$\begin{aligned} x_j^{(n+1)} &= \sigma\, \bar{x}_j^{-(n+1)}, \\ y_j^{(n+1)} &= \sigma\, y_j^{-(n+1)}, \\ z_j^{(n+1)} &= \sigma\, \bar{z}_j^{-(n+1)}, \\ u_j^{(n+1)} &= \sigma_u \bar{u}_j^{-(n+1)}, \\ v_j^{(n+1)} &= \sigma_v \bar{v}_j^{-(n+1)}, \\ w_j^{(n+1)} &= \sigma_w \bar{w}_j^{-(n+1)}. \end{aligned} \tag{12}$$

There are four quantities σ, σ_u, σ_v, and σ_w to be determined by the four conditions (5) and (8).

First we substitute (12) into (8) to get the σ_u, σ_v, and σ_w expressed by known quantities and the unknown quantity σ; then we use (5) to determine σ and the scheme will be complete.

Denote

$$\begin{aligned} XV &= \sum_{j=0}^{2} M_j \bar{x}_j^{-(n+1)} \bar{v}_j^{-(n+1)}, \\ XW &= \sum_{j=0}^{2} M_j \bar{x}_j^{-(n+1)} \bar{w}_j^{-(n+1)}, \\ YU &= \sum_{j=0}^{2} M_j \bar{y}_j^{-(n+1)} \bar{u}_j^{-(n+1)}, \\ YW &= \sum_{j=0}^{2} M_j \bar{y}_j^{-(n+1)} \bar{w}_j^{-(n+1)}, \\ ZU &= \sum_{j=0}^{2} M_j \bar{z}_j^{-(n+1)} \bar{u}_j^{-(n+1)}, \\ ZV &= \sum_{j=0}^{2} M_j \bar{z}_j^{-(n+1)} \bar{v}_j^{-(n+1)}. \end{aligned} \tag{13}$$

Putting (12) into (8) one gets:

$$\begin{aligned} \sigma_v(-ZV) + \sigma_w(YW) &= a/\sigma, \\ \sigma_u(ZU) + \sigma_w(-XW) &= b/\sigma, \\ \sigma_u(-YU) + \sigma_v(XV) &= c/\sigma. \end{aligned} \tag{14}$$

Hence one gets

$$\sigma_u = H_u/\sigma, \qquad \sigma_v = H_v/\sigma, \qquad \sigma_w = H_w/\sigma, \tag{15}$$

with

$$H_u = \begin{bmatrix} a & -ZV & YW \\ b & 0 & -XW \\ c & XV & 0 \end{bmatrix} \Bigg/ \begin{bmatrix} 0 & -ZV & YW \\ ZU & 0 & -XW \\ -YU & XV & 0 \end{bmatrix},$$
$$H_v = \begin{bmatrix} 0 & a & YW \\ ZU & b & -XW \\ -YU & c & 0 \end{bmatrix} \Bigg/ \begin{bmatrix} 0 & -ZV & YW \\ ZU & 0 & -XW \\ -YU & XV & 0 \end{bmatrix}, \tag{16}$$
$$H_w = \begin{bmatrix} 0 & -ZV & a \\ ZU & 0 & b \\ -YU & XV & c \end{bmatrix} \Bigg/ \begin{bmatrix} 0 & -ZV & YW \\ ZU & 0 & -XW \\ -YU & XV & 0 \end{bmatrix}.$$

Substituting (15) in (12), and then substituting (12) in (5), one gets the quadratic equation to determine σ:

$$\mathrm{Kin}(H_u\bar{u},\, H_v\bar{v},\, H_w\bar{w})^{(n+1)}/\sigma^2 + V(\bar{x}, \bar{y}, \bar{z})^{(n+1)}/\sigma - E = 0. \tag{17}$$

In this section we have assumed $E \geqslant 0$. Notice that in actual computations,

$$\mathrm{Kin}^{(n+1)} > 0, \qquad V^{(n+1)} < 0;$$

one can get the the unique positive root:

$$\frac{1}{\sigma} = \frac{[-V(\bar{x}, \bar{y}, \bar{z}) + \{V^2(\bar{x}, \bar{y}, \bar{z}) + 4E\,\mathrm{Kin}(H_u\bar{u},\, H_v\bar{v},\, H_w\bar{w})\}^{(1/2)}]^{(n+1)}}{2\mathrm{Kin}(H_u\bar{u},\, H_v\bar{v},\, H_w\bar{w})^{(n+1)}}. \tag{18}$$

Now the scheme runs in the order (9), (13), (16), (18), (15), and (12). All processes are explicit and all 10 identities, i.e., the ten algebraic integrals, are conserved.

The final results are as follows:

$$\begin{aligned} x_j^{(n+1)} &= \sigma\, \bar{x}_j^{-(n+1)}, \\ y_j^{(n+1)} &= \sigma\, \bar{y}_j^{-(n+1)}, \\ z_j^{(n+1)} &= \sigma\, \bar{z}_j^{-(n+1)}, \\ u_j^{(n+1)} &= (H_u/\sigma)\, \bar{u}_j^{-(n+1)}, \\ v_j^{(n+1)} &= (H_v/\sigma)\, \bar{v}_j^{-(n+1)}, \\ w_j^{(n+1)} &= (H_w/\sigma)\, \bar{w}_j^{-(n+1)}. \end{aligned} \tag{19}$$

The order of the scheme is:

(1) from (9), (13), and (16) to get H_u, H_v, and H_w;
(2) from (18) to get σ;
(3) from (9) and (19) to get the quantities in the $(n+1)$ step;

and one explicit cycle is complete.

Since there is no explicit analytic solution for comparison with numerical computation, one standard is that the correction factors σ, σ_u, σ_v, and σ_w should be as near unity as possible; the differences of these factors with 1 may serve as a measure of the errors. If no errors need to be corrected, then they all should be 1.

The following is an example of computation: Take the Lagrange triangular configuration as a base, double its initial positions and initial velocities to get $E>0$, and give some perturbations in the initial values to make this model not a planar motion. The initial values taken are as follows:

j	M	x	y	z	u	v	w
0	$\sqrt{3}$	0	2	0.01	-2	0	0.01
1	$\sqrt{3}$	$-\sqrt{3}$	-1	0.01	1	$-\sqrt{3}$	-0.02
2	$\sqrt{3}$	$\sqrt{3}$	-1	-0.02	1	$\sqrt{3}$	0.01

From these initial values one gets:

$$a = 0.141961524,$$
$$b = -0.141961524,$$
$$c = 20.7846096,$$
$$V = -2.59801126,$$
$$\text{Kin} = 10.3928245,$$
$$E = 7.79481321 > 0.$$

Again take the time increment $\tau = 0.001$, one can calculate this model for $t = n\tau$ and obtain the error indication quantities σ, σ_u, σ_v, and σ_w as shown in Table II. Hence we see that the values σ, σ_u, σ_v, and σ_w are quite near 1 as one should expect.

TABLE II

n	σ	σ_u	σ_v	σ_w
0	1	1	1	1
10	0.999980693	1.00001926	1.00001926	1.00001926
20	0.999973459	1.00002649	1.00002649	1.00002649
30	0.999973845	1.00002611	1.00002611	1.00002611
40	0.999974237	1.00002571	1.00002571	1.00002571
50	0.999974636	1.00002532	1.00002531	1.00002532
60	0.999975039	1.00002491	1.00002491	1.00002491
70	0.999975450	1.00002450	1.00002450	1.00002450
80	0.999975866	1.00002408	1.00002408	1.00002408
90	0.999976284	1.00002366	1.00002366	1.00002366
100	0.999976713	1.00002324	1.00002324	1.00002324
110	0.999977143	1.00002281	1.00002281	1.00002281
120	0.999977577	1.00002237	1.00002237	1.00002237
130	0.999978017	1.00002193	1.00002193	1.00002193
140	0.999978459	1.00002147	1.00002147	1.00002149
150	0.999978906	1.00002104	1.00002104	1.00002104
160	0.999979354	1.00002060	1.00002060	1.00002060
170	0.999979808	1.00002014	1.00002014	1.00002014
180	0.999980262	1.00001969	1.00001969	1.00001969
190	0.999980719	1.00001923	1.00001923	1.00001923
200	0.999981178	1.00001877	1.00001877	1.00001877

4. Remarks

(1) There is no difficulty in extending the method in Sections 2 and 3 to the n-body problem, only the terms in V and Kin should be increased accordingly, but there are still 10 algebraic integrals to be conserved. The amount of numerical calculations increases proportional to the number n.

(2) For $E < 0$, the method of Section 3 may give imaginary values for σ, so the time step should be limited in this case and a check of the sign of the quantity in the square root should be imposed accordingly.

(3) The formulas are symmetrical with respect to x, y, and z, but the initial values may not be symmetrical with respect to x, y, and z. For example, the initial position of the Lagrange triangular configuration is not symmetrical with respect to both x-axis and y-axis; this leads to the unsymmetrical results.

(4) In case the determinant of the system (14) is zero, further conditions are needed to solve the system. For example, the planar motion with $z \equiv 0$; then only one angular momentum must be considered.

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